Calogero-Vasiliev Oscillator in Dynamically Evolving Curved Spacetime

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Abstract

In a recent work, the consequences of quantizing a real scalar field Φ according to generalized "quon" statistics in a dynamically evolving curved spacetime (which, prior to some initial time and subsequent to some later time, is flat) were considered. Here a similar calculation is performed; this time we quantize Φ via the Calogero-Vasiliev oscillator algebra, described by a real parameter $\nu > -1/2$. It is found that both conservation ($\nu \to \nu$) and anticonservation ($\nu \to -\nu$) of statistics is allowed. We find that for mathematical consistency the Bogoliubov coefficients associated with the i'th field mode must satisfy $|\alpha_i|^2 - |\beta_i|^2 = 1$ with $|\beta_i|^2$ taking an integer value.

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I. INTRODUCTION

Particle creation in an expanding universe is a well studied aspect of quantum field theory in curved spacetime. Much of the original research in this area was performed by Parker [1–3]; his papers concern the interaction of a quantized field Φ with the classical background gravitational field arising from a spacetime dynamic between arbitrary times t_1 and t_2. As a particular example, we will choose the spatially flat ($n+1$) dimensional Robertson–Walker universe with metric

$$ds^2 = dt^2 - R^2(t)\delta_{ab}dx^a dx^b \qquad a, b = 1, 2, \ldots, n. \tag{1}$$

with constants R_1 and R_2 such that $R(t < t_1) = R_1$, $R(t > t_2) = R_2$. (The spacetime portions with $t < t_1$ and $t > t_2$ are usually referred to as the in- and out-regions respectively). Parker originally demonstrated [2] that if Φ was a real scalar field of spin zero obeying the Klein–Gordon equation; then if commutation relations were imposed upon the creation and annihilation operators in the in-region, it necessarily followed that commutation relation must be imposed upon the creation and annihilation operators in the out-region. It is not consistent to impose anti-commutation relations upon such a field Φ.

Parker later generalized the proof to spin 1/2 fermions [3] which obey the Dirac equation and satisfy anticommutation relations. The cases of higher spin bose/fermi fields, parastatistics [4] and ghost fields [5] have also been treated. Importantly, it was stressed that the flat space spin-statistics theorem could be inferred by taking the physical limit in which the dynamical part of the evolution becomes constant, and demanding that the commutation relations remain continuous in this limit.

Recently, an attempt was made to generalize this work to "quon" statistics [6], a case which interpolates between conventional Bose and Fermi statistics. ("Quons" are of great interest to theoretical physicists and have many suggested applications, for example see [7–12]). It was discovered that physically consistent evolution of the "quon" algebra from the in-region to the out-region took place only when the "quon" algebra reduced to the special cases of the Bose/Fermi algebras.

It is now natural to ask if Parker's work may be applied to other extensions of the Bose/Fermi algebra. Within this paper, we attempt to quantize Φ via the Calogero–Vasiliev [13,14] algebra

$$[a, a^\dagger] = 1 + 2\nu K \tag{2}$$

where $[A, B] = AB - BA$, K is related to the number operator by $K = (-1)^N$ and ν is a real parameter satisfying $\nu > -1/2$. We note further that N is given explicitly in terms of the creation and annihilation operators by

$$N = \frac{\{a, a^\dagger\}}{2} - \frac{(2\nu + 1)}{2} \tag{3}$$

where $\{A, B\} = AB + BA$; we shall also require the existence of a vacuum state $|0>$ such that $a|0>= 0$.

Relation (2) is an obvious generalization of the Bose algebra - it arises naturally in the quantization of the simple harmonic oscillator experiencing an additional inverse square potential - and has been the subject of much interest recently (for example, see [15–20]). In particular Macfarlane [21,22] has explored the links between the Calogero–Vasiliev system and parastatistics of order $p = 2\nu + 1$, potentially of great use since (2) is a bilinear relation as opposed to the trilinear relations usually associated with parastatistics. It is these links that motivate us to study the Calogero–Vasiliev oscillator in a dynamically evolving curved spacetime.

The structure of this paper then is as follows. In section II, we examine the evolution of a field Φ which is decomposed in terms of in- and out-region creation and annihilation operators, where each individual mode satisfies relation (2). (We will assume that different modes commute). We find that this can be performed in a physically consistent way provided that the usual Bose condition $|\alpha_i|^2 - |\beta_i|^2 = 1$ must is satisfied for each mode, where α_i and β_i are the (diagonal) Bogoliubov coefficients linking the in- and out-region operators. In section III, we obtain the unexpected result that $|\beta_i|^2$ must be an integer. Concluding remarks are given in section IV.

II. CALOGERO–VASILIEV OSCILLATOR IN DYNAMICALLY EVOLVING CURVED SPACETIME

As stated within the introduction, we will impose upon the in-region the following commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij}(1 + 2\nu K_i) \tag{4}$$

$$[a_i, a_j] = 0 \tag{5}$$

where $K_i = (-1)^{N_i^{in}}$ and ν is a real parameter which defines the statistics of the in-region; different modes are assumed to commute. We shall also assume that $\nu \neq 0$ since if that were the case, relations (4) and (5) would be the canonical commutation relations describing bosons - Parker's original work [2] would then force us to impose Bose statistics upon the out-region.

We may now decompose the field operator $\Phi(x^\mu)$ (where $\mu = 0, 1, 2, \ldots n$, $x^0 = t$) in terms of the in-region creation and annihilation operators obtaining

$$\Phi(x^\mu) = \sum_i [F_i(x^\mu)a_i + F_i^*(x^\mu)a_i^\dagger] \tag{6}$$

where we regard $\Phi(x^\mu)$ as a real scalar field within the Heisenberg picture (since there will be nothing inherently relativistic about this calculation, a similar treatment could be given for the Schrödinger field). $\{F_i(x^\mu)\}$ is a complete set of positive frequency solutions to the Klein–Gordon equation.

A similar expansion may also be performed in the out-region:

$$\Phi(x^\mu) = \sum_i [G_i(x^\mu)b_i + G_i^*(x^\mu)b_i^\dagger] \tag{7}$$

where $\{G_i(x^\mu)\}$ is also a set of positive frequency solutions to the Klein–Gordon equation and the operators b_i and $b_i^\dagger$ are in general different from the a_i and $a_i^\dagger$ due to particle creation in the expanding universe. We take the out-region algebra to be

$$[b_i, b_j^\dagger] = \delta_{ij}(1 + 2\omega L_i) \tag{8}$$

$$[b_i, b_j] = 0 \tag{9}$$

with the real parameter ω but not necessarily equal to ν, and $L_i = (-1)^{N_i^{out}}$. We note that $\omega \neq 0$ since otherwise one could reverse the direction of time in Parker's original work to show that this would force $\nu = 0$. Since both sets $\{F_i(x^\mu)\}$ and $\{G_i(x^\mu)\}$ are assumed to be complete, we may expand one in terms of the other viz:

$$G_i(x^\mu) = \sum_j [\alpha_{ij} F_j(x^\mu) + \beta_{ij} F_j^*(x^\mu)] \tag{10}$$

where the complex numbers α_{ij}, β_{ij} are known as Bogoliubov coefficients. However, for spacetimes whose metric is of the form (1), the Bogoliubov coefficients are known to be diagonal. Hence, we restrict ourselves to

$$\alpha_{ij} = \delta_{ij}\alpha_i \qquad \beta_{ij} = \delta_{ij}\beta_i \tag{11}$$

Using relations (6), (7), (10) and (11) one obtains the relation between the in- and out-region operators:

$$a_i = \alpha_i b_i + \beta_i^* b_i^\dagger \tag{12}$$

We now substitute (12) and its hermitian conjugate into relation (4). After using the defining relations for the out-region statistics we obtain

$$(|\alpha_i|^2 - |\beta_i|^2)(1 + 2\omega L_i) = (1 + 2\nu K_i). \tag{13}$$

If it were not for the operator terms involving L_i and K_i, this would be the usual Bose condition $|\alpha_i|^2 - |\beta_i|^2 = 1$. However, we shall show that (13) is in fact this condition in disguise. We take the scalar product of (13) with the out-region vacuum state $|0, out>$ which satisfies $b_i|0, out>= 0$. We find that

$$(|\alpha_i|^2 - |\beta_i|^2)(1 + 2\omega) = 1 + 2\nu < 0, out|K_i|0, out> \tag{14}$$

It becomes necessary at this point to calculate the value of $< 0, out|N_i^{in}|0, out >$. Using (3) and the transformation (12) we may rewrite the in-region Number operator as

$$N_i^{in} = (|\alpha_i|^2 + |\beta_i|^2)[N_i^{out} + \frac{(2\omega + 1)}{2}] + \alpha_i\beta_i b_i b_i + \alpha_i^*\beta_i^* b_i^\dagger b_i^\dagger - \frac{(2\nu + 1)}{2}. \tag{15}$$

Thus,

$$< 0, out|N_i^{in}|0, out >= (|\alpha_i|^2 + |\beta_i|^2)\frac{(2\omega + 1)}{2} - \frac{(2\nu + 1)}{2}. \tag{16}$$

Crucially, this is a real number. Therefore for equation (14) to remain entirely real (since we assume $\nu \neq 0$), $< 0, out|K_i|0, out >$ may only take the values of plus or minus one. Hence, we have two alternatives:

$$< 0, out|K_i|0, out >= +1 \quad \Rightarrow \quad |\alpha_i|^2 - |\beta_i|^2 = \frac{1 + 2\nu}{1 + 2\omega} \tag{17a}$$

$$< 0, out|K_i|0, out >= -1 \quad \Rightarrow \quad |\alpha_i|^2 - |\beta_i|^2 = \frac{1 - 2\nu}{1 + 2\omega} \tag{17b}$$

We may also take the scalar product of (13) with the normalized 1-particle state $|1_i, out >\sim b_i^\dagger|0, out >$. We obtain

$$(|\alpha_i|^2 - |\beta_i|^2)(1 - 2\omega) = 1 + 2\nu < 1_i, out|K_i|1_i, out > \tag{18}$$

where, again, to keep everything real, we must have that $< 1_i, out|K_i|1_i, out >= \pm 1$. By adding (18) to (14) we discover that we have 3 possibilities:

$$< 0, out|K_i|0, out >=< 1_i, out|K_i|1_i, out >= +1 \quad \Rightarrow \quad |\alpha_i|^2 - |\beta_i|^2 = 1 + 2\nu \tag{19a}$$

$$< 0, out|K_i|0, out >=< 1_i, out|K_i|1_i, out >= -1 \quad \Rightarrow \quad |\alpha_i|^2 - |\beta_i|^2 = 1 - 2\nu \tag{19b}$$

$$< 0, out|K_i|0, out >= - < 1_i, out|K_i|1_i, out > \quad \Rightarrow \quad |\alpha_i|^2 - |\beta_i|^2 = 1 \tag{19c}$$

We may now rule out two of these possibilities. If we compare (17a) with (19a), we find that $\omega = 0$. This contradicts one of our original assumptions, and so we discard this alternative. Similarly, comparing (17b) with (19b) also gives $\omega = 0$.

Our only remaining option then is to adopt $|\alpha_i|^2 - |\beta_i|^2 = 1$. Even then we have two choices; we may adopt $< 0, out|K_i|0, out >= +1$; then from comparing (19c) with (17a) we conclude that $\nu = \omega$ i.e conservation of statistics or we may adopt $< 0, out|K_i|0, out >= -1$; then comparing (19c) with (17b) we would conclude that $\nu = -\omega$ i.e. anticonservation of statistics.

III. CONSERVATION AND ANTICONSERVATION OF STATISTICS

We now examine the situation more carefully for the case when $\nu = \omega$. Consider the scalar product of the normalized n-particle states $|n_i, out >\sim (b_i^\dagger)^n|0, out >$.

$$< n_i, out|N_i^{in}|n_i, out > = (|\alpha_i|^2 + |\beta_i|^2)[n + \frac{(2\nu + 1)}{2}] - \frac{(2\nu + 1)}{2} \tag{20}$$

$$= n + |\beta_i|^2(2n + 2\nu + 1) \tag{21}$$

where relation (21) follows from (15) and we have used the relation $|\alpha_i|^2 - |\beta_i|^2 = 1$ in obtaining (21). We now use (21) in two special cases:

$$< 0, out|K_i|0, out >= +1 \quad \Rightarrow \quad (-1)^{|\beta_i|^2(2\nu+1)} = +1 \tag{22}$$

$$< 1_i, out|K_i|1_i, out >= -1 \quad \Rightarrow \quad (-1)^{1+|\beta_i|^2(2\nu+3)} = -1 \tag{23}$$

Substituting (22) into (23) we obtain

$$(-1)^{2|\beta_i|^2} = +1 \tag{24}$$

We deduce from this that $|\beta_i|^2$ must be an integer. An almost identical calculation reveals that $|\beta_i|^2$ must be an integer when the statistics are anticonserved i.e. when $\nu = -\omega$.

A direct consequence of this result is that in either case it is not possible to take the physical limit in which the dynamical part of the evolution goes to zero; hence for this model the connection between spin and statistics in flat space may not be inferred.

IV. DISCUSSION

We have shown that if a real scalar field $\Phi(x^\mu)$ is quantized according to relations (4) and (5), then the particle statistics may either be conserved or anticonserved upon evolution from the in-region to the out-region (depending on whether $< 0, out|K_i|0, out >$ is an even or odd integer respectively). The usual Bose relation between the Bogoliubov coefficients must hold for each field mode i.e. $|\alpha_i|^2 - |\beta_i|^2 = 1$. It is also necessary that $|\beta_i|^2$ must be an integer for the evolution of the statistics to take place in a physically acceptable way.

It is worth commenting upon the following form for the in-region statistics (a generalization to which relations (4) and (5) belong):

$$[a_i, a_j^\dagger] = \delta_{ij} F(N_i^{in}) \quad , \quad [a_i, a_j] = 0 \tag{25}$$

where $F(r)$ is a function which returns a real number when the real variable r is an integer, and returns a complex number when r is non-integer. We assume the number operator N_i^{in} to be of the form

$$N_i = \frac{\{a_i, a_i^\dagger\}}{2} + const. \tag{26}$$

for a spacetime whose metric is of the form of relation (1). Similar relations are supposed for the out-region:

$$[b_i, b_j^\dagger] = \delta_{ij} G(N_i^{out}) \quad , \quad [b_i, b_j] = 0 \tag{27}$$

where G is a similar function to F (but not necessarily equal), and we wish to evolve the in-region statistics to those of the out-region in a physically consistent way. The in-region and out-region creation/annihilation operators are still linked by (12). Taking the scalar product with the out-region vacuum state and the 1-particle state $|1_i, out >$ yields

$$(|\alpha_i|^2 - |\beta_i|^2)G(0) = F(< 0, out|N_i^{in}|0, out >) \tag{28}$$

$$(|\alpha_i|^2 - |\beta_i|^2)G(1) = F(< 1_i, out|N_i^{in}|1_i, out >) \tag{29}$$

In this case (since $G(0)$ and $G(1)$ are real numbers), to keep relations (28) and (29) real both of the following must be satisfied:

$$< 0, out|N_i|0, out > \in \{0, 1, 2, 3, 4, 5, 6, 7, 8......\} \tag{30}$$

$$< 1_i, out|N_i|1_i, out > \in \{0, 1, 2, 3, 4, 5, 6, 7, 8......\} \tag{31}$$

This in turn requires that $|\alpha_i|^2 + |\beta_i|^2$ must be an integer. Essentially this follows from chosen form of the number operator.

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